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Stability Analysis for a Class of Affine State-Space Neural Networks

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ABSTRACT

This paper presents some new analytical results on the global stability for a general class of discrete-time state-space neural networks by using the Lyapunov stability theory and the contraction mapping theorem. The proposed stability conditions are determined by the synaptic weights matrices of the neural network.

KEY WORDS

Affine state-space neural networks, nonlinear identification, Lyapunov stability, equilibrium states.

1. Introduction

The increasing necessity for modelling high complex systems has been a source of motivation towards the development of robust black-box methodologies to deal with nonlinear dynamics and uncertainty. Two main research directions have been pursued in the context of system's identification: one focusing on the synthesis of nonlinear model structures and the other centred on the development of efficient parameter estimation algorithms.

Among nonlinear model structures artificial neural networks (or simply neural networks) have attracted a great deal of interest mostly due to their inherent property of density, that is, they present the universal approximation property [1], [2], [3]. This means that given a sufficient number of processing units (also referred to as neurons) and at least one hidden layer, neural networks (specifically multilayer perceptrons) with sigmoidal activation functions can effectively approximate to any level of accuracy a given nonlinear function. In the case of neural networks with internal feedback connections one can prove as well (see e.g. [4] or [5]) they provide universal identification model structures on compact subsets of the state and input spaces and assuming finite time intervals.

The incorporation of MLP structures in nonlinear system identification has independently been proposed by several authors, namely, Chen et al. [6] and Narendra and Parthasarathy [7]. The rationale behind this idea is that the internal system dynamics can appropriately be reconstructed form a finite data set of sampled inputs and outputs, as shown by F. Takens [8]. Since this neural network topology is exclusively static, that is, it has only the ability to approximate spatial mappings, the incorporation of temporal information is indirectly carried out by providing to the network (as an extended input vector) a sequence of past system's inputs and outputs. Despite this approach has proven successful in many applications (see e.g. [9]) it presents the inextricable drawback of being quite sensitive to the lag window [10]. Another well reported weakness is related to its intrinsic susceptibility to noise corrupted input and output signals.

When feedback connections are established within the hidden layer of a threelayered neural network not only the model becomes less vulnerable to noise, since the input vector provided to the network does not contain past outputs but the whole internal system's history may intrinsically be encapsulated into the model's dynamics. Furthermore, as pointed out in [11], non-linear statespace neural predictors constitute a broader class of nonlinear dynamical models and demand a smaller number of parameters, which are to be adjusted in the training stage.

For the very reason that a state-space neural network structure defines a dynamical system it is crucial to investigate its fundamental properties in terms of dynamics (stability, periodic orbits, convergence) and bifurcation. These issues have all been the focus of a significant research activity in the last few years so as to bring some light into the characterization of recurrent neural network architectures' dynamic behaviour (see e.g. [12], [13], [14]).

In this paper it is proposed a general class of recurrent neural networks with intrinsic potential application to nonlinear system's identification. Given its structure, the neural predictor can be described in the state-space form by a set of nonlinear difference equations. Globally asymptotic stability conditions of the equilibrium in the restrict sense of a constant input (step response) are derived for this affine state-space neural network structure.

2. Neural Network Topology

The general class of discrete-time dynamic neural networks considered in this work comprises three layers, as depicted in Fig.1. Since this recurrent neural network is here regarded as a nonlinear model structure in the context of system identification the input and output layers must incorporate as much neurons as the number of the inputs and outputs of the system to be modelled. In what the hidden layer is concerned, this particular architecture consists of neurons presenting two types of activation functions, namely, sigmoidal activation functions and linear activation functions. The number of neurons to be incorporated within the hidden layer should be selected in order to achieve an optimal approximation which is reflected on a good neural predictor's generalization. Actually, a network larger than might be expected may result in considerable specialization or overfitting, while a deficient number of neurons may not be feasible to represent appropriately the non-linear mapping underlying the sampled input-output data.

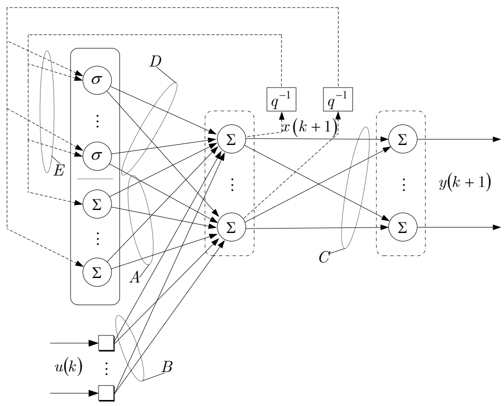

Fig.1. Statespace neural network structure.

As can be observed from Fig.1 the proposed neural network architecture has in a topological sense hybrid features which are provided by the two kinds of activation functions (sigmoidal and linear) found within the hidden layer. The incorporation of these two types of neurons is known to improve the neural predictor's modelling performance in case of mild nonlinear dynamics of the system to be identified [15].

The affine nonlinear discrete-time system can be described in the statespace form as

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + D\sigma(Ex(k)) \\ y(k) = Cx(k) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the neural statespace vector, $y \in \mathbb{R}^p$ the neural output and $u \in \mathbb{R}^m$ the input vector. A, B, C, D and E are real-valued matrices of appropriate dimensions. The nonlinear activation function $\sigma(\cdot)$ is here a continuous and differentiable sigmoidal function satisfying the following conditions: $\sigma(\cdot)$ is upper and lower bounded; $\lim_{t\to\pm\infty} \sigma(t) = \pm 1$; $\sigma(t) = 0 \Leftarrow t = 0$; $\sigma'(t) > 0$, $\lim_{t\to\pm\infty} \sigma'(t) = 0$ and $\max(\sigma'(t)) \le 1$, at $t = 0$.

3. Equilibrium Points

Consider the general affine nonlinear system (1) with $\sigma(\cdot) = \tanh(\cdot)$. Given an initial state vector x_0 and an input sequence $U \triangleq \{u(0),\dots,u(N-1)\}$ provided to the system the state trajectory can be expressed at any time k as

$$\begin{aligned} x(k) &= A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} \left\{ Bu(j) + D\sigma\left[Ex(j)\right] \right\} \\ &\triangleq \phi(k, x_0, \{u(i)\}_{i=0}^{k-1}) \end{aligned} \tag{2}$$

where $\phi(\cdot)$ denotes the state-transition function. An important issue concerning the stability of the state trajectory is to investigate what happens to the system's behaviour when $k \to \infty$ and the input sequence is held constant. Thus by studying possibly equilibrium points one can conclude upon the stability or instability of the system under consideration.

Definition 1: A point $x \in \mathbb{R}^n$ is defined as an equilibrium point of a given (nonlinear) system if $x(k+1) = x(k), \forall k > \tau \in \mathbb{N}$.
Concerning the system (1) and assuming $car(I-A) = n$ this implies that

$$x = (I-A)^{-1}\left[Bu + D\sigma(Ex)\right] \triangleq g(x) \tag{3}$$

Furthermore, these equilibrium points are fixed points of function $g(x)$.

Theorem 1: Let $W = W(A,B,D,E)$ be a given parameterisation for the statespace neural network (1) and assume $car(I-A) = n$ and $u(k) \in \mathcal{U} \subseteq \mathbb{R}^m$. Then there exist at least one equilibrium point in the hypercube Ω defined as

$$\Omega = \left\{ x : \left| x - (I-A)^{-1} Bu \right| \le \frac{\sqrt{n}}{\rho(I-A)} \|D\|_2 \right\} \tag{4}$$

where $|\cdot|$ means the Euclidean norm, $\|M\|_2 \triangleq \left(\lambda_{\max}\left(M^{\mathrm{T}}M\right)\right)$ is the induce matrix norm, and $\rho(\cdot)$ the spectral radius.

The proof follows immediately from the Brouwer's fixed point theorem (see e.g. [16]).

4. Global Stability Results

Definition 2 (Globally Asymptotically Stability): The neural network structure (1) is said to be global asymptotically stable if it has a unique stable equilibrium point $x_e \in \mathbb{R}^n$ and its attractive region is the whole space $\mathbb{R}^n$ for any finite constant input sequence $u(i) \equiv \bar{u}$ such that $x_e = \lim_{k\to+\infty} \phi(k, x_0, \bar{u})$.

The globally asymptotically stability (GAS for short) concept implicitly points out to a feasible approach so as to investigate the stability of a given neural network parameterisation on the basis of the Euclidean norm. In fact, the asymptotic convergence to a given equilibrium point implies the underlying operator contractiveness.

Lemma 1 (Global Contraction Mapping) (*in* [17]): Let $(X, \|\cdot\|)$ be a Banach space and consider $\Psi : X \mapsto X$. Suppose there exists a real-valued constant $\rho < 1$ such that

$$\|\Psi x_1 - \Psi x_2\| \le \rho \|x_1 - x_2\|, \forall x_1, x_2 \in X$$

Then there exists exactly one fixed point x^* such that $x^* = \Psi x^*$. Moreover, for any $x_0 \in X$ the sequence $\{x_n\}$ defined by $x_{n+1} = \Psi x_n$ converges to x^*.

Theorem 2: Let $W = W(A,B,D,E)$ be a given parameterisation for the statespace neural network (1) and assume $\varepsilon \triangleq \max|e_{ij}|$ where e_{ij} is an entry of matrix E. Then for each $u(k) = \overline{u} \in \mathcal{U} \subseteq \mathbb{R}^m$ the neural network is contractive and has one equilibrium point if, and only if

$$\|A\|_2 + \varepsilon \|D\|_2 < 1$$

Proof: Let start by transforming the neural network (1) into a new coordinate system where the origin is centred on an equilibrium point. The new state vector is thus defined by

$$z(k) = x(k) - x_e \tag{5}$$

Taking into account (5) the statespace neural can be expressed as

$$\begin{aligned} z(k+1) = Az(k) + B\big(u(k) - \overline{u}\big) + \\ +D\Big\{\sigma\big[E\big(z(k) - x_e\big)\big] - \sigma\big(Ex_e\big)\Big\} \end{aligned} \tag{6}$$

Without loss of generality let assume $u(k) \equiv \overline{u}$, then equation (6) can be written as

$$\begin{aligned} z(k+1) = Az(k) + \\ +D\Big\{\sigma\big[E\big(z(k) - x_e\big)\big] - \sigma\big(Ex_e\big)\Big\} \end{aligned} \tag{7}$$

Now defining a continuous nonlinear function $\phi : \mathbb{R}^n \mapsto \mathbb{R}^n$ as

$$\phi(z) = D\Big\{\sigma\big[E\big(z(k) - x_e\big)\big] - \sigma\big(Ex_e\big)\Big\} \tag{8}$$

such that $\phi(0) = 0$, equation (7) can be rewritten in the following form:

$$z(k+1) = Az(k) + D\phi(z) \tag{9}$$

The above expression describes an autonomous system dynamics for which $z \equiv 0$ represents the equilibrium point.

Now by applying the Lagrange mean value theorem to the nonlinear part of (1) follows

$$\big|\sigma(Ex_1) - \sigma(Ex_2)\big| \le \varepsilon |x_1 - x_2| \tag{10}$$

with $\varepsilon \triangleq \max|e_{ij}|$.

For two arbitrary vectors z_1 and z_2 the following inequality holds:

$$\begin{aligned} \big|\phi(z_1) - \phi(z_2)\big| &= \Big|\sigma\big(E(z_1 + x_e)\big) - \sigma\big(E(z_2 + x_e)\big)\Big| \\ &\le \varepsilon |z_1 - z_2| \end{aligned} \tag{11}$$

Now defining $\psi(z) \triangleq z(k+1) = Az(k) + D\phi(z(k))$, yields

$$\begin{aligned} \big|\psi(z_1) - \psi(z_2)\big| &= \Big|A(z_1 - z_2) + D\big(\phi(z_1) - \phi(z_2)\big)\Big| \\ &\le \|A\|_2 |z_1 - z_2| + \|D\|_2 \big|\phi(z_1) - \phi(z_2)\big| \\ &\le \big(\|A\|_2 + \varepsilon\|D\|_2\big)|z_1 - z_2| \end{aligned} \tag{12}$$

According to lemma 1 the neural network will have one asymptotically stable equilibrium point if, and only if,

$$\|A\|_2 + \varepsilon \|D\|_2 < 1 \tag{13}$$

Another possible approach to investigate the neural system's stability makes use of the Lyapunov theory. If one can find a candidate Lyapunov function $\mathcal{V}(\cdot)$ for which $\Delta\mathcal{V}(\cdot) \le 0$ the system's stability is immediately proved.

Let assume as a candidate Lyapunov function the following function:

$$\mathcal{V}(\cdot) = z^{\mathrm{T}}(k) P z(k) \tag{14}$$

where $P \in \mathbb{R}^{n \times n}$ and $z(k)$ as in (9).

Theorem 3: Let $W = W(A,B,D,E)$ be a given parameterisation for the state space neural network (1). Then the equilibrium point is globally asymptotically stable if there exists a diagonal positive matrix P and a positive definite matrix Q satisfying $\overline{A}^{\mathrm{T}} P \overline{A} + \varepsilon^2 \overline{D}^{\mathrm{T}} P \overline{D} - P = Q$, with $\overline{A} = \sqrt{2}A$, $\overline{D} = \sqrt{2}D$ and $\varepsilon \triangleq \max|e_{ij}|$.

Proof: Let first apply a coordinate transformation to (1) such that the origin is coincident with the equilibrium point. The new state equation can be written as in (9).

Next, assume a candidate Lyapunov function as given in (14) then,

$$\Delta\mathcal{V}(z)=\left[Az+D\phi(z)\right]^{\mathrm{T}}P\left[Az+D\phi(z)\right]- \\ -z^{T}Pz \tag{15}$$

or

$$\Delta\mathcal{V}(z)=z^{T}\left(A^{\mathrm{T}}PA-P\right)z+2z^{T}A^{\mathrm{T}}PD\phi(z)+ \\ +\phi^{\mathrm{T}}(z)D^{\mathrm{T}}PD\phi(z) \tag{16}$$

On the other hand,

$$\left[-Az+D\phi(z)\right]^{\mathrm{T}}P-\left[Az+D\phi(z)\right]\geq 0 \tag{17}$$

which implies

$$z^{\mathrm{T}}A^{\mathrm{T}}PAz+\phi^{\mathrm{T}}(z)D^{\mathrm{T}}PD\phi(z)\geq 2z^{\mathrm{T}}A^{\mathrm{T}}PD\phi(z) \tag{18}$$

Taking into account this inequality into (16), follows

$$\Delta\mathcal{V}(z)\leq z^{\mathrm{T}}\left(A^{\mathrm{T}}PA-P\right)z+z^{\mathrm{T}}A^{\mathrm{T}}PAz \\ +\phi^{\mathrm{T}}(z)D^{\mathrm{T}}PD\phi(z)+\phi^{\mathrm{T}}(z)D^{\mathrm{T}}PD\phi(z) \tag{19}$$

Since $\left|\phi(z)\right|\leq\left|\varepsilon z\right|$ then

$$\Delta\mathcal{V}(z)\leq z^{\mathrm{T}}\left(A^{\mathrm{T}}PA-P\right)z+z^{\mathrm{T}}A^{\mathrm{T}}PAz \\ +2\varepsilon^{2}z^{\mathrm{T}}D^{\mathrm{T}}PDz \tag{20}$$

or

$$\Delta\mathcal{V}(z)\leq z^{\mathrm{T}}\left[2A^{\mathrm{T}}PA-P+2\varepsilon^{2}D^{\mathrm{T}}PD\right]z \tag{21}$$

If there exist a positive definite matrix Q such that $\bar{A}^{\mathrm{T}}P\bar{A}+\varepsilon^{2}\bar{D}^{\mathrm{T}}P\bar{D}-P=-Q$, with $\bar{A}=\sqrt{2}A$ and $\bar{D}=\sqrt{2}D$, then the Lyapunov function is a monotone decreasing function $\left(\Delta\mathcal{V}(\cdot)\leq 0\right)$ which implies that the equilibrium point is globally asymptotically stable.

5. An Example

To illustrate the stability analysis procedure it is considered in this section the identification of a distributed solar collector field (DSC), part of the Plataforma Solar de Almería (Fig.2). This solar power plant is located in the desert of Tabernas in the South of Spain.

Fig.2. The distributed solar collector field.

The DSC field consists of 480 parabolic trough collectors arranged in 20 parallel rows aligned on West-East axis and forming 10 independent loops.

To capture the DSC's dynamics a neural network structure as that proposed in this work was considered and a training set collected from the plant (Fig.3), where T is the output oil temperature and $\dot{v}$ the oil flow rate.

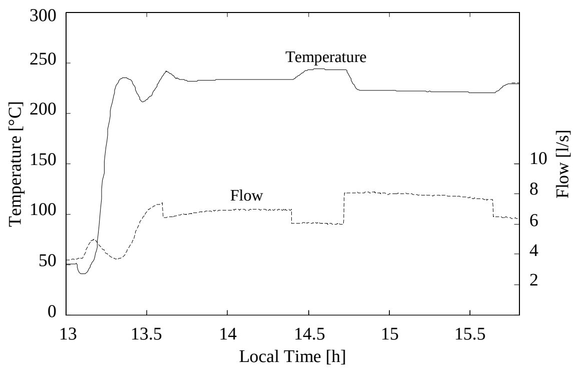

Fig.3. Training set.

After the training completed using the Levenberg-Marquardt algorithm the weights matrices took the following form, being E set as the identity matrix for convenience:

$$A=\begin{bmatrix}8.74\times10^{-1} & -6.05\times10^{-3}\\ 1.21\times10^{-2} & 8.58\times10^{-1}\end{bmatrix};\ B=\begin{bmatrix}8.81\times10^{-2}\\ -1.42\times10^{-2}\end{bmatrix}$$

$$C=\begin{bmatrix}1.08\\ -2.86\times10^{-2}\end{bmatrix}^{\mathrm{T}};D=\begin{bmatrix}1.71\times10^{-2} & 4.42\times10^{-2}\\ -6.17\times10^{-2} & -4.98\times10^{-2}\end{bmatrix}$$

In order to test the stability of this neural preditor let evaluate the stability condition (13).

$$\|A\|_2 + \varepsilon \|D\|_2 = 0.9647 < 1$$

According to theorem 2 the neural model has for each constant input sequence one asymptotically globally stable equilibrium point.

6. Conclusions

A new class of affine neural networks is proposed in this paper, which is more general than other recurrent structures such as the Hopfield neural network, making them a serious candidate for nonlinear system identification. Based on the Lyapunov stability theory and the contraction mapping theorem some analytical conditions on the global asymptotically stability are presented.

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